11.6 Convergence Tests

1210. The Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series such that $0 < a_n \le b_n$ for all n.

- If $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is also convergent.
- If $\sum_{n=1}^{\infty} a_n$ is divergent then $\sum_{n=1}^{\infty} b_n$ is also divergent.

1211. The Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series such that a_n and b_n are positive for all n.

- If $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$ then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are either both convergent or both divergent.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ then $\sum_{n=1}^{\infty} b_n$ convergent implies that $\sum_{n=1}^{\infty} a_n$ is also convergent.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ then $\sum_{n=1}^{\infty} b_n$ divergent implies that $\sum_{n=1}^{\infty} a_n$ is also divergent.

1212. p-series

p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $0 < p \le 1$.

1213. The Integral Test

Let $f(x)$ be a function which is continuous, positive, and decreasing for all $x \ge 1$. The series

$$\sum_{n=1}^{\infty} f(n) = f(1) + f(2) + f(3) + \ldots + f(n) + \ldots$$

converges if $\int_1 f(x)dx$ converges, and diverges if

$\int_1^n f(x)dx \to \infty$ as $n \to \infty$.

1214. The Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.

- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ then $\sum_{n=1}^{\infty} a_n$ may converge or diverge and the ratio test is inconclusive; some other tests must be used.

1215. The Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.

- If $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$ then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$ then $\sum_{n=1}^{\infty} a_n$ may converge or diverge, but no conclusion can be drawn from this test.